Abstract

Lie groups arise as symmetries of differential geometric objects. Due to their ‘curved’ nature Lie groups are oftentimes hard to work with directly. Sophus Lie’s contribution to the then so called ‘transformation groups’ is the discovery that linearizing them retains a lot of information. This produces in a functorial fashion a Lie algebra. Lie theory, the study of this construction, plays the role for differential equations what Galois theory does for polynomial equations. However, this perspective is not well-understood. The first quarter of the course deals with necessary preliminaries while the second and the third quarters deal with Lie groups and Lie algebras, respectively. The last quarter deals with their representation theories. For a fixed vector space, its group of linear automorphisms is a Lie group while its ring of endomorphisms has a natural Lie algebra structure. These two can be regarded as a ‘concrete’ Lie group and Lie algebra, respectively. The goal of representation theory is to turn abstractly defined Lie groups and Lie algebras into concrete ones. Not surprisingly, the relation between a Lie group and its Lie algebra lifts to a nice relation between their representation theories. As a foresight, the course will be guided by the interest in Lie groups. Then Lie algebras will be introduced to easily facilitate the study of such groups. Finally, representation theory is used to study Lie algebras in a more concrete way.

Course Outline

I. Preliminaries
   1. Topological spaces, smooth structures, and vector fields
   2. Differentiation and smooth maps
   3. Categories and functors

II. Lie Groups
   4. Definitions and examples
   5. The topology of Lie groups
   6. Subgroups and homomorphisms
   7. Coverings of Lie groups
III. The Lie Functor

8. Lie algebra of a Lie group
9. Functoriality
10. The exponential map
11. Matrix Lie groups and their Lie algebras
12. The adjoint representation
13. Subgroups and subalgebras
15. Lie’s Third Theorem

IV. Lie Algebras

16. Simple, solvable and nilpotent Lie algebras
17. Semisimple Lie algebras
18. Simultaneous diagonalization and Cartan subalgebras
19. Root systems
20. Classification of semisimple Lie algebras

V. Representation Theory

21. Definitions and examples
22. Differentiating and integrating representations
23. Weights
24. Verma modules
25. The universal enveloping algebra
26. Completeness of characters and Fourier analysis

References

